

$\zeta(n)$ via hyperbolic functions.

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Abstract

We present here an approach to a computation of $\zeta(2)$ by changing variables in the double integral using hyperbolic trig functions. We also apply this approach to present $\zeta(n)$, when $n > 2$, as a definite improper integral of single variable.

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1 Introduction

The Riemann zeta function is defined as the series

$$\zeta(n) = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \dots + \frac{1}{k^n} + \dots$$

for any integer $n \geq 2$. Three centuries ago Euler found that $\zeta(2) = \pi^2/6$, which is an irrational number. Exact value of $\zeta(3)$ is still unknown though it was proved by Apéry in 1979 that $\zeta(3)$ was also irrational (see [5]). Values of $\zeta(n)$, when n is even, are known and can be written in terms of Bernoulli numbers. We refer the interested reader to chapter 19 of [1] for a “perfect” proof of the formula

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} B_{2k}}{(2k)!} \cdot \pi^{2k} \quad (k \in \mathbb{N}).$$

Notice that $\zeta(n)$ can be written as the following multi-variable integral

$$\zeta(n) = \int_0^1 \dots \int_0^1 \frac{1}{1 - x_1 x_2 \dots x_n} dx_1 dx_2 \dots dx_n.$$

Indeed, each integral is improper at both ends and since the geometric series $\sum_{q \geq 0} x^q$ converges uniformly on the interval $|x| \leq R$, $\forall R \in (0, 1)$ we can write

$$\frac{1}{1 - x_1 x_2 \cdots x_n} = \sum_{q=0}^{\infty} (x_1 x_2 \cdots x_n)^q$$

then interchange summation with integration, and then integrate $(x_1 x_2 \cdots x_n)^q$ for each q . Using the identities

$$\frac{1}{1 - xy} + \frac{1}{1 + xy} = \frac{2}{1 - x^2 y^2} \quad \text{and} \quad \frac{1}{1 - xy} - \frac{1}{1 + xy} = \frac{2xy}{1 - x^2 y^2}$$

and a simple change of variables one can easily see that

$$\int_0^1 \int_0^1 \frac{1}{1 - xy} dx dy = \frac{4}{3} \int_0^1 \int_0^1 \frac{1}{1 - x^2 y^2} dx dy$$

by further generalizing this idea one comes along the following,

$$\zeta(n) = \frac{2^n}{2^n - 1} \int_0^1 \cdots \int_0^1 \frac{1}{1 - \prod_{i=1}^n x_i^2} dx_1 \dots dx_n.$$

Notice that $(1, 1)$ is the only point in the square $[0, 1] \times [0, 1]$, which makes the integrand $1/(1 - x^2 y^2)$ singular. If we take another point on the graph of $1 = x^2 y^2$, say $(a, 1/a)$ with $a \in (0, \infty)$, then it follows easily (see lemma 1 below) that

$$\int_0^{1/a} \int_0^a \frac{1}{1 - x^2 y^2} dx dy = \int_0^1 \int_0^1 \frac{1}{1 - x^2 y^2} dx dy.$$

This result motivates the following definition

Definition 1. For any point $(a_1, a_2, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$ such that $a_i \in (0, +\infty)$, $\forall i \in \{1, \dots, n-1\}$ we define

$$I_n(a_1, \dots, a_{n-1}) = \int_0^{\frac{1}{a_1 \cdots a_{n-1}}} \cdots \int_0^{a_2} \int_0^{a_1} \frac{1}{1 - \prod_{i=1}^n x_i^2} dx_1 dx_2 \dots dx_n.$$

Lemma 1. For any $a_i \in (0, +\infty)$, we have $I_n(a_1, \dots, a_{n-1}) = I_n(1, 1, \dots, 1)$.

Proof. Simply observe that by using the change of variables $x_i = a_i u_i$ for all $i \in [1, \dots, n]$, where $a_n = 1/(a_1 a_2 \cdots a_{n-1})$, the Jacobian equals 1, and the integrand is unchanged. \square

In this article we investigate $\zeta(n)$ following Beukers, Calabi and Kolk (see [2]), who used the change of variables

$$x = \frac{\sin(u)}{\cos(v)} \quad \text{and} \quad y = \frac{\sin(v)}{\cos(u)} \quad \text{to evaluate} \quad \int_0^1 \int_0^1 \frac{1}{1 - x^2 y^2} dx dy.$$

Such proof of the identity $\zeta(2) = \pi^2/6$ may also be found in chapter 6 of [1] and in papers of Elkies [3] and Kalman [4]. Let us also mention here that Kalman's paper, in addition to a few other proofs of the identity, contains some history of the problem together with an extensive reference list.

Here we will be changing variables too, but in the integrals $I_n(a_1, \dots, a_{n-1})$ and using the hyperbolic trig functions \sinh and \cosh instead of \sin and \cos . Such a change of variables was considered independently of us by Silagadze and the reader will find his results in [6].

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2 Hyperbolic Change of Variables

First observe that the change of variables

$$x_i = \frac{\sin(u_i)}{\cos(u_{i+1})} \quad \forall i \in \mathbb{N} \mod (n)$$

reduces the integrand in $I_n(1, \dots, 1)$ to 1 only when n is even. The region of integration $\Phi_n = [(x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_1, \dots, x_n < 1]$ becomes the one-to-one image of the n -dimensional polytop (note $u_{n+1} = u_1$)

$$\Pi_n := [(u_1, u_2, \dots, u_n) \in \mathbb{R}^n : u_i > 0, u_i + u_{i+1} < \frac{\pi}{2}, 1 \leq i \leq n].$$

We suggest here a different change of variables that will produce an integrand of 1 for all values of n in $I_n(a_1, \dots, a_{n-1})$. But first we define the corresponding region.

Definition 2. For any point $(a_1, a_2, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$ such that $a_i \in (0, +\infty)$, $\forall i \in \{1, \dots, n-1\}$ we define

$$\Phi_n(a_1, a_2, \dots, a_{n-1}) := [(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 < x_i < a_i, \forall i \in \{1, \dots, n\}],$$

where $a_n = 1/(a_1 \cdot a_2 \cdot \dots \cdot a_{n-1})$.

Lemma 2. The change in variables

$$x_i = \frac{\sinh(u_i)}{\cosh(u_{i+1})} \quad \forall i \in \mathbb{N} \pmod{n}$$

reduces the integrand of $I_n(a_1, \dots, a_{n-1})$ to 1 for all values of $n \geq 2$. It also gives a one-to-one differentiable map between the region $\Phi_n(a_1, a_2, \dots, a_{n-1})$ and the set $\Gamma_n \subset \mathbb{R}^n$ described by the following n inequalities:

$$0 < u_i < \operatorname{arcsinh}(a_i \cdot \cosh(u_{i+1})), \quad \forall i \in \mathbb{N} \pmod{n}.$$

Proof. The inequalities for Γ_n follow trivially from the corresponding inequalities $0 < x_i < a_i$ and the facts that $\cosh(x) > 0$ and $\operatorname{arcsinh}(x)$ is increasing everywhere. Injectivity and smoothness of the map may be proven by writing down formulas, which express each u_i in terms of all x_j . For example, here are the corresponding formulas for the set Γ_3 :

$$u_i = \operatorname{arcsinh} \left(x_i \cdot \sqrt{\frac{1 + x_{i+1}^2 + x_{i-1}^2 x_{i+1}^2}{1 - x_1^2 x_2^2 x_3^2}} \right), \quad i \in \mathbb{N} \pmod{3}.$$

The Jacobian is the determinant of the matrix

$$A = \begin{pmatrix} \frac{\cosh(u_1)}{\cosh(u_2)} & \frac{-\sinh(u_1) \sinh(u_2)}{\cosh^2(u_2)} & 0 & \dots & 0 \\ 0 & \frac{\cosh(u_2)}{\cosh(u_3)} & \frac{-\sinh(u_2) \sinh(u_3)}{\cosh^2(u_3)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-\sinh(u_n) \sinh(u_1)}{\cosh^2(u_1)} & 0 & 0 & \dots & \frac{\cosh(u_n)}{\cosh(u_1)} \end{pmatrix}$$

To compute this determinant we observe that the first column expansion reduces the computation to two determinants of the upper and lower triangular matrices. This results in the formula, where the first term comes from the upper triangular matrix and the second from the lower triangular matrix (recall that $u_{n+1} = u_1$) :

$$\text{Det}(A) = \prod_{i=1}^n \frac{\cosh(u_i)}{\cosh(u_{i+1})} + (-1)^{n-1} \cdot \prod_{i=1}^n \frac{-\sinh(u_i) \sinh(u_{i+1})}{\cosh^2(u_{i+1})} = 1 - \prod_{i=1}^n \tanh^2(u_i).$$

When using the above change in variables the denominator of the integrand $1 - \prod_{i=1}^n x_i$ becomes $1 - \prod_{i=1}^n \tanh^2(u_i)$, which we just proved to be the Jacobian. \square

3 Computations of $\zeta(2)$

We begin with $\zeta(2)$, which is a rational multiple of $I_2(1)$. Lemma 1 implies that it's enough to compute

$$I_2(a) = \int_0^{\frac{1}{a}} \int_0^a \frac{1}{1 - x^2 y^2} dx dy \quad \text{for arbitrary } a > 0.$$

We now perform the following change in variables

$$x = \frac{\sinh(u)}{\cosh(v)}, \quad y = \frac{\sinh(v)}{\cosh(u)}.$$

As we proved above, our integrand reduces to 1 and all we must do is worry about the limits. If $x = 0$ then clearly $u = 0$, the same is true for y and v . If $x = a$ then $a \cdot \cosh(v) = \sinh(u)$ so $v = \text{arccosh}(\frac{\sinh(u)}{a})$ and if $y = \frac{1}{a}$ then $(1/a) \cdot \cosh(u) = \sinh(v)$ so $v = \text{arcsinh}(\frac{\cosh(u)}{a})$ thus describing our region of integration (see Figure 1). We then write the integral $I_2(a)$ as follows

$$\int_0^{\text{arcsinh}(a)} \text{arcsinh}\left(\frac{\cosh(u)}{a}\right) du + \int_{\text{arcsinh}(a)}^{\infty} \text{arcsinh}\left(\frac{\cosh(u)}{a}\right) - \text{arccosh}\left(\frac{\sinh(u)}{a}\right) du.$$

Lemma 3. $\lim_{a \rightarrow 0} \int_0^{\text{arcsinh}(a)} \text{arcsinh}\left(\frac{\cosh(u)}{a}\right) du = 0$

Proof. If we let $\cosh(\text{arcsinh}(z)) = Q$ then $Q = \sqrt{1 + z^2}$. Therefore

$$\text{arcsinh}\left(\frac{\cosh(\text{arcsinh}(a))}{a}\right) = \text{arcsinh}\left(\sqrt{\frac{1}{a^2} + 1}\right).$$

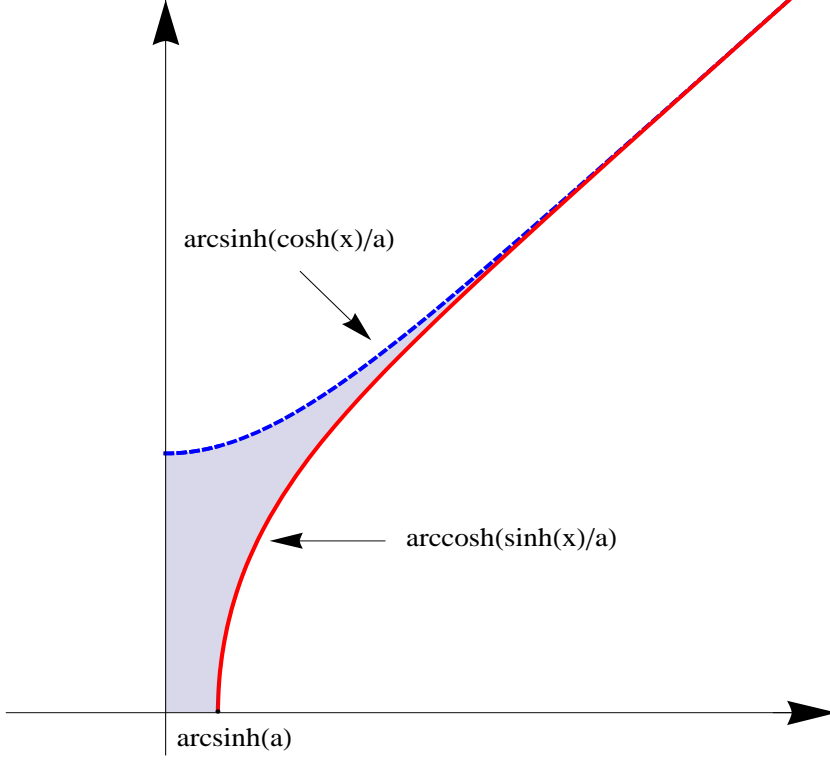


Figure 1: The set $\Gamma_2 \subset \mathbb{R}^2$, $\forall a > 0$.

Since $\text{arcsinh}(\cosh(u)/a)$ is concave up, we can take area of the rectangle with vertices at $(0, 0)$, $(\text{arcsinh}(a), 0)$, and $(\text{arcsinh}(a), \text{arcsinh}(\cosh(\text{arcsinh}(a))/a))$ as an overestimate of the integral, that is

$$\text{arcsinh}(a) \cdot \text{arcsinh}\left(\sqrt{\frac{1}{a^2} + 1}\right) \geq \int_0^{\text{arcsinh}(a)} \text{arcsinh}\left(\frac{\cosh(u)}{a}\right) du \geq 0$$

Then by applying L'hospital's rule one can deduce

$$\lim_{a \rightarrow 0} (\text{arcsinh}(a) \cdot \text{arcsinh}\left(\sqrt{\frac{1}{a^2} + 1}\right)) = 0.$$

□

Now, since $I_2(a) = I_2(1)$, $\forall a > 0$, we conclude that $I_2(1) = \lim_{a \rightarrow 0} I_2(a)$, and therefore we have

$$I_2(1) = \lim_{a \rightarrow 0} \int_{\text{arcsinh}(a)}^{\infty} \text{arcsinh}\left(\frac{\cosh(u)}{a}\right) - \text{arccosh}\left(\frac{\sinh(u)}{a}\right) du.$$

Since

$$\operatorname{arcsinh}\left(\frac{\cosh(x)}{a}\right) = \ln\left(\frac{\cosh(x)}{a} + \sqrt{\frac{\cosh^2(x)}{a^2} + 1}\right)$$

and

$$\operatorname{arcosh}\left(\frac{\sinh(x)}{a}\right) = \ln\left(\frac{\sinh(x)}{a} + \sqrt{\frac{\sinh^2(x)}{a^2} - 1}\right)$$

we get

$$I_2(1) = \lim_{a \rightarrow 0} \int_{\operatorname{arcsinh}(a)}^{\infty} \ln\left(\frac{\frac{\cosh(x)}{a} + \sqrt{\frac{\cosh^2(x)}{a^2} + 1}}{\frac{\sinh(x)}{a} + \sqrt{\frac{\sinh^2(x)}{a^2} - 1}}\right) dx$$

which, after taking the limit as $a \rightarrow 0$ gives

$$I_2(1) = \int_0^{\infty} \ln\left(\frac{\cosh(x)}{\sinh(x)}\right) dx.$$

Using integration by parts $u = \ln\left(\frac{\cosh(x)}{\sinh(x)}\right)$ and $v = dx$ one obtains the formula

$$I_2(1) = x \ln\left(\frac{\cosh(x)}{\sinh(x)}\right) \Big|_0^{\infty} + \int_0^{\infty} \frac{2x}{\sinh(2x)} dx.$$

By examining the limits of the first half of the formula as x goes to 0 and ∞ we are left with only the integral

$$I_2(1) = \int_0^{\infty} \frac{2x}{\sinh(2x)} dx.$$

By applying the change in variables $u = 2x$ our formula becomes

$$I_2(1) = \frac{1}{2} \int_0^{\infty} \frac{u}{\sinh(u)} du.$$

Now we use the method of differentiation under the integral sign and consider the function

$$F(\alpha) = \frac{1}{2} \int_0^{\infty} \frac{\operatorname{arctanh}(\alpha \tanh(x))}{\sinh(x)} dx.$$

One should consider the function F at the points $\alpha = 1$ and $\alpha = 0$. $F(1)$ is clearly the integral we are trying to find and $F(0)$ is 0. Thus by differentiating under the integral with respect to α , plus some algebra we obtain

$$F'(\alpha) = f(\alpha) = \frac{1}{2} \int_0^{\infty} \frac{\cosh(x)}{1 + (1 - \alpha^2)\sinh^2(x)} dx.$$

Then by performing the change in variables $u = \sqrt{1 - \alpha^2} \cdot \sinh(x)$ the integral becomes

$$f(\alpha) = \frac{1}{2\sqrt{1 - \alpha^2}} \int_0^\infty \frac{1}{1 + u^2} du,$$

which is simply

$$\frac{\operatorname{arctanh}(u)}{2\sqrt{1 - \alpha^2}} \Big|_0^\infty = \frac{\pi}{4\sqrt{1 - \alpha^2}}.$$

Since we took the derivative with respect to α we must take the integral with respect to α so we have

$$\int_0^1 f(\alpha) d\alpha = F(1) - F(0) = F(1) - 0 = F(1)$$

which, as stated above is our goal. So

$$I_2(1) = \int_0^1 f(\alpha) d\alpha = \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1 - \alpha^2}} d\alpha = \frac{\pi}{4} \operatorname{arcsin}(\alpha) \Big|_0^1 = \frac{\pi^2}{8},$$

and hence $\zeta(2) = \frac{4}{3} \cdot \frac{\pi^2}{8} = \pi^2/6$.

4 A formula for $\zeta(n)$, $n \geq 2$

One could try to use similar approach to compute $\zeta(n)$, $n > 2$, however the computations become a bit long. Instead, we present an elementary proof of the following theorem, which generalizes our formula for $\zeta(2)$ from the previous section.

Theorem. *Let $n \geq 2$ be a natural number. Then*

$$\int_0^1 \dots \int_0^1 \frac{1}{1 - \prod_{i=1}^n x_i^2} dx_1 \dots dx_n = \frac{1}{(n-1)!} \cdot \int_0^\infty \ln^{n-1}(\coth(x)) dx.$$

Let us start with the following lemma, which can be easily proved by using induction on k , integration by parts and l'Hospital's rule.

Lemma 4.

$$\int_0^1 \ln^k(z) z^{2q} dz = \frac{(-1)^k k!}{(2q+1)^{k+1}}, \quad \forall k \in \mathbb{N} \text{ and } q \geq 0.$$

Proof of the theorem. Applying the substitution $z = \tanh(x)$ to the integral

$$\frac{1}{(n-1)!} \int_0^\infty \ln^{n-1}(\coth(x)) \, dx$$

gives

$$\frac{1}{(n-1)!} \int_0^1 \frac{(-\ln(z))^{n-1}}{1-z^2} dz = \frac{1}{(n-1)!} \int_0^1 (-\ln(z))^{n-1} \cdot \left(\sum_{q \geq 0} z^{2q} \right) dz.$$

Since the integral is improper at both ends and the geometric series $\sum_{q \geq 0} z^{2q}$ converges uniformly on the interval $|z| \leq R$, $\forall R \in (0, 1)$, the last integral equals

$$\frac{1}{(n-1)!} \sum_{q \geq 0} (-1)^{n-1} \cdot \int_0^1 \ln^{n-1}(z) z^{2q} dz = \text{by lemma 4} = \sum_{q \geq 0} \frac{1}{(2q+1)^n}.$$

Using the geometric series expansion one can easily show that we also have

$$\int_0^1 \cdots \int_0^1 \frac{1}{1 - \prod_{i=1}^n x_i^2} dx_1 \cdots dx_n = \sum_{q \geq 0} \frac{1}{(2q+1)^n}.$$

□

Corollary. For any integer $n \geq 2$,

$$\zeta(n) = \frac{2^n}{(2^n - 1) \cdot (n-1)!} \cdot \int_0^\infty \ln^{n-1}(\coth(x)) \, dx.$$

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